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A STEADY SOLUTION OF THE VLASOV EQUATION

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There are two basic techniques for measurement of ion temperature (T) in the Ionosphere: radar scattering using powerful ground stations and in situ probing using ion-traps on board satellites. Recently traps have been used on board rockets at the bottom of the Ionosphere |1| and in other planetary ionospheres |2|. A trap is a multigrid electrostatic probe |3|. An entrance grid E at an opening in the spacecraft wall repels incoming electrons (the potential V_E, like that of the satellite V_s, is negative relative to the potential in the undisturbed plasma). Entering ions strike a collector C. Between E and C there is a retarding grid P, biased at a positive value V_p, which rejects the less energetic ions. The current to C is registered as a function of V_p.

Characteristic lengths of the problem are the plasma mean free path ($\sim 10^4$ cm), satellite size ($\sim 10^2$ cm), opening width (~ 10 cm), distance between grids (~ 1 cm), and Debye length λ_D (~ 1 cm). Outside a sheath of thickness λ_D next to wall and opening the plasma must be quasineutral |3|. The spacecraft velocity U is large (small) compared with the ion (electron) thermal speed. An ideal analysis (electric distrubances confined to the sheath, onedimensional flow through sheath and inside trap, grids characterized by just optical transparencies), leads to Whipple's formula |4|, used to interpret measurements.

Note that ions rejected by P leave E as a high-velocity jet so that the disturbance produced by the trap is not entirely confined to the sheath |5|. We analyse here the mathematical problem arising from this phenomenon; a correction to Whipple's formula may then be obtained. We take the satellite wall be the (infinite) plane $x = 0$, the grid E being a circle of radius R centered at $y = z = 0$; the plasma fills the half-space $x < 0$. For simplicity let $V_s = V_E$, and the velocity U be normal to the wall. In a frame moving with the spacecraft the potential field V and the ion distribution function $N_o f/Z$ will obey steady Vlasov and Poisson equations

$$\bar{v} \cdot \nabla f - \frac{Ze}{m} \nabla V \cdot \frac{\partial f}{\partial \bar{v}} = 0 \tag{1}$$

$$\nabla^2 V = 4\pi e N_o \left[\exp\left(\frac{eV}{T_e}\right) - \int f \, d\bar{v} \right] \tag{2}$$

(Z is ion charge number, N_o and T_e electron density and temperature); Boltzmann's law is valid for $e|V_s|/T_e$ large except near the wall where the electron density is unimportant.

Since there are no collisions, ion trajectories would be straight were it not for the electric field $-\nabla V$. It will then prove convenient to partition f in the form

$$f \equiv f^+ + f^-, \quad f^+(v_x < 0) \equiv 0, \quad f^-(v_x > 0) \equiv 0.$$

Boundary condtions are then

$$V \to 0, \ f^+(v_x > 0) \to f_\infty(v_x - U, \bar{v}_\perp) \quad \text{as} \quad x \to -\infty, \tag{3}$$

$$V = V_E, \ f^-(v_x < 0) = H(R - r_\perp) g(\bar{v}, \bar{r}_\perp) \quad \text{at} \quad x = 0. \tag{4}$$

H is Heavyside function and $f_\infty(\bar{v})$ the ion distribution in the far away undisturbed plasma. Note that all ions striking the wall are absorbed. The function g is a characteristic of the trap; for the ideal condtions of Whipple's formula,

$$g = \alpha_E^2 H \left\{ \left[2eZ \frac{V_p - V_E}{m} \right]^{1/2} - |v_x| \right\} f^+(x = 0, \bar{r}_\perp, \bar{v}_\perp, |v_x|), \quad v_x < 0 \tag{5}$$

α_E being the transparency of E.

Consider α_E^2 small and write $V = V_o + \alpha_E^2 V_1 + \ldots$, $f^\pm = f_o^\pm + \alpha_E^2 f_1^\pm + \ldots$ To lowest order the problem is onedimensional. If the potential is monotonic, as in fact it will be, we obviously have $f_o^- \equiv 0$, since $V_E < 0$. Then

$$\frac{d^2 V_o}{dx^2} = 4\pi e N_o \left[\exp\left(\frac{eV_o}{T_e}\right) - \int dv_x \int d\bar{v}_\perp f_o^+ \right] \tag{6}$$

$$v_x \frac{\partial f_o^+}{\partial x} - \frac{Ze}{m} \frac{dV_o}{dx} \frac{\partial f_o^+}{\partial v_x} = 0. \tag{7}$$

Using (3), the solution to (7) is

$$f_o^+ = f_\infty \left[\{ v_x^2 + \frac{2eZ}{m} V_o(x) \}^{1/2} - U, \bar{v}_\perp \right]. \tag{8}$$

If f is characterized by just a thermal energy T, as in a Maxwellian distribution, Eqs. (6) and (8) yield V_o/V_E as a function of x/λ_D, eV_E/T_e, T/ZT_e, and Mach number $M \equiv (mU^2/T)^{1/2}$; here, $\lambda_D \equiv (4\pi e^2 N_o/T_e)^{1/2}$. V_o/V_E goes to zero in a distance of order λ_D [$\int f_\infty d\bar{v} = 1$ within terms exponentially small, $\sim\exp(-M^2/2)$].

To first order in α_E^2 we linearize Eqs. (1) and (2). Outside the sheath perturbations will be quasineutral:

$$0 = \frac{eV_1}{T_e} - \int f_1^+ d\bar{v} - \int f_1^- d\bar{v}. \tag{9}$$

For f_1^+ we have

$$\left\{v_x \frac{\partial}{\partial x} + \bar{v}_\perp\cdot\frac{\partial}{\partial \bar{r}_\perp}\right\} f_1^+ = \frac{Ze}{m}\left\{\frac{\partial V_1}{\partial x}\,\frac{\partial}{\partial v_x} + \frac{\partial V_1}{\partial \bar{r}_\perp}\cdot\frac{\partial}{\partial \bar{v}_\perp}\right\} f_\infty, \quad v_x > 0. \tag{10}$$

We next introduce Fourier transforms with respect to $\bar{r}_\perp$ [e.g. $\tilde{V}(\bar{K},x) \equiv \int d\bar{r}_\perp V(\bar{r}_\perp,x)\exp(-i\bar{K}\cdot\bar{r}_\perp)$]. Using $f_1^+(x\to-\infty)\to 0$, we solve (10) and introduce $\tilde{f}_1^+$ in (9) to get

$$\frac{e\tilde{V}_1}{T_e} = \int \tilde{f}_1^- d\bar{v} + \int_{-\infty}^{x} dx'\left\{\frac{e\tilde{V}_1(x')}{T_e} L_1(x-x') + \frac{e}{T_e}\,\frac{\partial \tilde{V}_1}{\partial x'} L_2(x-x')\right\} \tag{11}$$

where

$$L_1(S) = \int \frac{d\bar{v}}{v_x}\,\frac{ZT_e}{m}\, i\bar{K}\cdot\frac{\partial f_\infty}{\partial \bar{v}_\perp}\exp\left\{\frac{-i\bar{K}\cdot\bar{v}_\perp}{v_x} S\right\}, \tag{12a}$$

$$L_2(S) = \int \frac{d\bar{v}}{v_x}\,\frac{ZT_e}{m}\,\frac{\partial f_\infty}{\partial v_x}\exp\left\{\frac{-i\bar{K}\cdot\bar{v}_\perp}{v_x} S\right\}. \tag{12b}$$

Since $f_0^- \equiv 0$, the equation for f_1^- outside the sheath is

$$\left\{v_x \frac{\partial}{\partial x} + \bar{v}_\perp\cdot\frac{\partial}{\partial \bar{r}_\perp}\right\} f_1^- = 0,$$

and therefore

$$\tilde{f}_1^-(x) = \tilde{f}_1^-(o)\exp\left\{\frac{-i\bar{K}\cdot\bar{v}_\perp}{v_x} x\right\}. \tag{13}$$

The spectral width of $\tilde{f}_1^-$ can be seen to be $\Delta K \sim 1/R$. Hence, the characteristic length of the region outside the sheath, where the disturbance spreads because of the thermal motion, is of order of MR. The meaning of $\tilde{f}_1^-(o)$ in (13) is then $\lim \tilde{f}_1^-(x/MR)$ as $x/MR \to 0$. To find this limit we must analyse the sheath to first order. The equation for f_1^- is

$$\left\{v_x \frac{\partial}{\partial x} + \bar{v}_\perp\cdot\frac{\partial}{\partial \bar{r}_\perp}\right\} f_1^- - \frac{Ze}{m}\,\frac{dV_0}{dx}\,\frac{\partial f_1^-}{\partial v_x} = 0; \tag{14}$$

the second term is smaller than the first by a factor λ_D/MR. Dropping it, Eq. (14) is solved using (4) and (5)

$$f_1^- = H(R-r_\perp)\, H\left\{\left[2eZ\,\frac{V_p - V_o(x)}{m}\right]^{1/2} - |v_x|\right\} f_\infty\left\{\left[v_x^2 + \frac{2ZeV_o(x)}{m}\right]^{1/2} - U, \bar{v}_\perp\right\}$$

Taking the limit $x/\lambda_D \to -\infty$, Fourier-transforming with respect to $\bar{r}_\perp$ and matching to the outer region we get

$$\tilde{f}_1^-(o) = \frac{2\pi R}{K} J_1(KR)\, H\left\{\left(\frac{2eZV_p}{m}\right)^{1/2} - |v_x|\right\} f_\infty\left\{|v_x| - U, \bar{v}_\perp\right\}, \quad v_x < 0. \tag{15}$$

Using (12), (13), and (15) in (11) leads to an integral equation defined only in a half-space, singular, and with a difference kernel, as in the Wiener-Hopf problem. The present equation is of the Volterra type, however. To solve it, we consider an extended problem: to determine a function $\tilde{V}_1(x)$ that satisfies (11) for all real x. This requires that the inhomogeneous term [call it $\tilde{g}(x)$] of Eq. (11) be given for $x > 0$ too. We now write

$$\tilde{g}(x<0) \equiv \int \tilde{f}_1^- d\bar{v}, \quad \tilde{g}(x>0) \equiv \tilde{h}(x)$$

where $\tilde{h}(x)$ is supposed to be known. Then introducing Fourier transforms with respect to x [$\tilde{\tilde{V}}_1(P) \equiv \int_{-\infty}^{\infty} dx\, \exp(i Px)\tilde{V}_1(x)$, and so on] we get $\tilde{\tilde{V}}_1$, and by Fourier inversion,

$$\frac{e\tilde{V}_1}{T_e} = \int_{-\infty}^{\infty}\frac{dP}{2\pi}\,\frac{\exp(-iPx)}{1+L(P)}\left\{\int_{-\infty}^{0} dx' e^{-ix'P}\int \tilde{f}_1^- d\bar{v} + \int_0^{\infty} dx' e^{-ix'P}\tilde{h}(x')\right\}, \tag{16}$$

$$L(P) \equiv -\int_0^{\infty} ds\, e^{iPs}\left(L_1(s) - iP\, L_2(s)\right).$$

Equation (16), for $x<0$, solves our original problem whatever $\tilde{h}(x)$ may be (provided that the integrals do exist). For the solution to be unique however the contribution from $\tilde{h}(x')$ to (16), for $x<0$, should vanish independently of $\tilde{h}(x')$, a condition equivalent to

$$\int_{-\infty}^{\infty} \frac{dP}{2\pi} \frac{e^{iP(x'-x)}}{1+L(P)} \equiv 0, \quad x'-x>0. \tag{17}$$

This will be the case if $L(P)$, when continued analytically for complex P, is analytical in the upper half-plane, and if $1+L(P)=$ $=0$ has no roots there; then the integral (17), rewritten as

$$\frac{1}{1+L_+(\infty)}\delta(x'-x)+\int_{-\infty}^{\infty}\frac{dP}{2\pi}\frac{[L_+(\infty)-L(P)]\exp[iP(x'-x)]}{[1+L_+(\infty)][1+L(P)]}, \quad x'-x>0,$$

will clearly vanish $[L_+(\infty)\equiv L(|P|\to\infty,\ I_mP>0)]$.

To understand the above conditions note that the dispersion relation for longitudinal waves in an homogeneous plasma with ion distribution f_∞ (taking the z-axis along $\bar{K}$) reads

$$1-\frac{ZT_e}{m}\int\frac{dv_z}{v_z-\frac{\omega}{K}}\frac{d}{dv_z}\int f_\infty\, dv_x\, dv_y=0, \quad \text{for } I_m\omega>0, \tag{18}$$

for ion-acoustic (quasineutral, low-frequency) waves |6|. Now, to lowest order in an expansion in powers of M^{-2} we find

$$L(P)=-\frac{ZT_e}{m}\int\frac{dv_z}{v_z-\frac{UP}{K}-i\delta}\frac{d}{dv_z}\int f_\infty\, dv_x\, dv_y, \quad \delta\to 0^+.$$

Under very weak restrictions $L(P)$ is analytical in the upper half-space, where $L\sim P^{-2}$ for $|P|\to\infty$. Clearly the equation $1+L=0$ has no roots with $I_mP>0$ if Eq. (18) has no roots with $I_m\omega>0$, that is, <u>if the undisturbed plasma is stable in the Vlasov sense against quasineutral, low-frequency perturbations.</u>

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